Real Analysis

the roots of unity in $K$, then by (2.8.b)

by the above, they all have the same

in $\hat{G}$ consists of $z \in (\mathbb{R}^s \times \mathbb{C}^t)^*$ such that

$< N(z) \leq 1$

$l(z) = \sum_{k=1}^{s+t} \xi_k l_k$

Restriction of automorphisms gives rise

$Gal(L/\mathbb{Q}[k]) \cong Gal(\hat{G})/Q$
Real Analysis

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December 4, 2016
Contents

1 Families of Sets 1
2 Measures 3
3 Construction of Measures 5
4 Measurable Functions 7
5 The Lebesgue Integral 9
6 Limit Theorems 11
7 Differentiation and Integration 13
8 Product Measures 17
9 Signed Measures 19
10 $L^p$ Spaces 21
11 The Radon-Nikodym Theorem 25
12 Linear Operators 27
13 Hilbert Spaces 31
Chapter 1

Families of Sets

Definition 1.0.1. An algebra is a collection $\mathcal{A}$ of subsets of $X$ such that
(i) $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$;
(ii) if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$;
(iii) if $A_1, \ldots, A_n \in \mathcal{A}$, then $\bigcup_{i=1}^n A_i$ and $\bigcap_{i=1}^n A_i$ are in $\mathcal{A}$.

$\mathcal{A}$ is a $\sigma$-algebra if in addition
(iv) whenever $A_1, A_2, \ldots$ are in $\mathcal{A}$, then $\bigcup_{i=1}^\infty A_i$ and $\bigcap_{i=1}^\infty A_i$ are in $\mathcal{A}$.

Since $\bigcap_{i=1}^\infty A_i = (\bigcup_{i=1}^\infty A_i)^c$, the requirement that $\bigcap_{i=1}^\infty A_i$ be in $\mathcal{A}$ is redundant. The pair $(X, \mathcal{A})$ is called a measurable space. A set $A$ is measurable or $A$ measurable if $A \in \mathcal{A}$.

Lemma 1.0.2. If $\mathcal{A}_\sigma$ is a $\sigma$-algebra for each $\sigma$ in some non-empty index set $I$, then $\bigcap_{\sigma \in I} \mathcal{A}_\sigma$ is a $\sigma$-algebra.

If $\mathcal{G}$ is the collection of open subsets of $X$, then we call $\sigma(\mathcal{G})$ the Borel $\sigma$-algebra on $X$, and this is often denoted $\mathcal{B}$. Elements of $\mathcal{B}$ are called Borel sets and are said to be Borel measurable. When $X = \mathbb{R}$, $\mathcal{B}$ is not equal to $2^\mathbb{R}$.

Proposition 1.0.3. If $X = \mathbb{R}$, then the Borel $\sigma$-algebra $\mathcal{B}$ is generated by each of the following collection of sets:
(i) $\mathcal{C}_1 = \{(a, b) : a, b \in \mathbb{R}\}$;
(ii) $\mathcal{C}_2 = \{[a, b] : a, b \in \mathbb{R}\}$;
(iii) $\mathcal{C}_3 = \{(a, b] : a, b \in \mathbb{R}\}$;
(iv) $\mathcal{C}_4 = \{(a, \infty) : a \in \mathbb{R}\}$.
Chapter 2

Measures

Definition 2.0.1. Let $X$ be a set and $\mathcal{A}$ a $\sigma$-algebra consisting of subsets of $X$. A measure on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$;
(ii) if $A_i \in \mathcal{A}$, $i = 1, 2, \ldots$, are pairwise disjoint, then

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

The second condition is known as countable additivity. We say a set function is finitely additive if $\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$ whenever $A_1, \ldots, A_n$ are in $\mathcal{A}$ and are pairwise disjoint. The triple $(X, \mathcal{A}, \mu)$ is called a measure space.

Proposition 2.0.2. The following hold:

(i) If $A, B \in \mathcal{A}$ with $A \subset B$, then $\mu(A) \leq \mu(B)$.
(ii) If $A_i \in \mathcal{A}$ and $A = \bigcup_{i=1}^{\infty} A_i$, then $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$.
(iii) Suppose $A_i \in \mathcal{A}$ and $A_i \uparrow A$. Then $\mu(A) = \lim_{n \to \infty} \mu(A_n)$.
(iv) Suppose $A_i \in \mathcal{A}$ and $A_i \downarrow A$. If $\mu(A_1) < \infty$, then we have $\mu(A) = \lim_{n \to \infty} \mu(A_n)$.

Definition 2.0.3. A measure $\mu$ is a finite measure if $\mu(X) < \infty$. A measure $\mu$ is $\sigma$-finite if there exist sets $E_i \in \mathcal{A}$ for $i = 1, 2, \ldots$ such that $\mu(E_i) < \infty$ for each $i$ and $X = \bigcup_{i=1}^{\infty} E_i$. If $\mu$ is a finite measure, then $(X, \mathcal{A}, \mu)$ is called a finite measure space, and similarly, if $\mu$ is a $\sigma$-finite measure, then $(X, \mathcal{A}, \mu)$ is called a $\sigma$-finite measure space.

Let $(X, \mathcal{A}, \mu)$ be a measure space. A subset $A \subset X$ is a null set if there exists a set $B \in \mathcal{A}$ with $A \subset B$ and $\mu(B) = 0$. We do not require $A$ to be in $\mathcal{A}$. If $\mathcal{A}$ contains all the null sets, then $(X, \mathcal{A}, \mu)$ is said to be a complete measure space. The completion of $\mathcal{A}$ is the smallest $\sigma$-algebra $\overline{\mathcal{A}}$ containing $\mathcal{A}$ such that $(X, \overline{\mathcal{A}}, \mu)$ is complete.
Chapter 3

Construction of Measures

Definition 3.0.1. Let $X$ be a set. An outer measure is a function $\mu^*$ defined on $2^X$ satisfying

(i) $\mu^*(\emptyset) = 0$;
(ii) if $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$;
(iii) $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$ whenever $A_1, A_2, \ldots$ are subsets of $X$.

A set $N$ is a Null set with respect to $\mu^*$ if $\mu^*(N) = 0$. A common way to generate outer measures is as follows.

Proposition 3.0.2. Suppose $C$ is a collection of subsets of $X$ such that $\emptyset$ and $X$ are both in $C$. Suppose $\ell : C \to [0, \infty]$ with $\ell(\emptyset) = 0$. Define

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \ell(A_i) : A_i \in C \text{ for each } i \text{ and } E \subset \bigcup_{i=1}^{\infty} A_i \right\}. \quad \text{Then } \mu^* \text{ is an outer measure.}$$

We will see that while $\mu^*$ is an outer measure, it is not a measure on $2^\mathbb{R}$. We will also see that if we restrict $\mu^*$ to a $\sigma$-algebra $\mathcal{L}$ which is strictly smaller than $2^\mathbb{R}$, then $\mu^*$ will be a measure on $\mathcal{L}$. In the above proposition, when $C$ is the collection of intervals, and $\ell$ is merely the length of an interval, then that measure is what is known as Lebesgue measure. The $\sigma$-algebra $\mathcal{L}$ is called the Lebesgue $\sigma$-algebra.

Definition 3.0.3. Let $\mu^*$ be an outer measure. A set $A \subset X$ is $\mu^*$-measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

for all $E \subset X$.

Theorem 3.0.4. (Carathéodory Theorem) If $\mu^*$ is an outer measure on $X$, then the collection $\mathcal{A}$ of $\mu^*$-measurable sets is a $\sigma$-algebra. If $\mu$ is the restriction of $\mu^*$ to $\mathcal{A}$, then $\mu$ is a measure. Moreover, $\mathcal{A}$ contains all the null sets.

From here on, we shall usually use ’m’ instead of $\mu$ when talking of Lebesgue measures.
Proposition 3.0.5. Every set in the Borel $\sigma$-algebra on $\mathbb{R}$ is $m^*$-measurable.

Proposition 3.0.6. If $e$ and $f$ are finite and $I = (e, f]$, then $m^*(I) = \ell(I)$.

In the special case of Lebesgue measure, the collection of $m^*$-measurable sets is called the **Lebesgue $\sigma$-algebra**. A set is **Lebesgue measurable** if it is in the Lebesgue $\sigma$-algebra.

The Lebesgue measure of a countable set is 0. However, there are uncountable sets which have Lebesgue measure 0, like the Cantor set.

The countable intersections of open sets are sometimes called $G_\delta$ sets; The countable unions of closed sets are called $F_\sigma$.

Theorem 3.0.7. Let $E$ be any set of real numbers. Then each of the following assertions is equivalent to the measurability of $E$.

(i) For each $\epsilon > 0$, there is an open set $O$ containing $E$ for which $m^*(O - E) < \epsilon$.

(ii) There is a $G_\delta$ set $G$ containing $E$ for which $m^*(G - E) = 0$.

(iii) For each $\epsilon > 0$, there is a closed set $F$ contained in $E$ for which $m^*(E - F) < \epsilon$.

(iv) There is an $F_\sigma$ set $F$ contained in $E$ for which $m^*(E - F) = 0$.

A measure on an algebra is called a **premeasure**.

Theorem 3.0.8. (Carathéodory Extension Theorem) Suppose $\mathcal{A}_0$ is an algebra and $\ell : \mathcal{A}_0 \to [0, \infty]$ is a premeasure on $\mathcal{A}_0$. Define

$$
\mu^*(E) = \inf \{\sum_{i=1}^{\infty} \ell(A_i) : each \ A_i \in \mathcal{A}_0, E \subset \bigcup_{i=1}^{\infty} A_i\}
$$

for $E \subset X$. Then

(i) $\mu^*$ is an outer measure;

(ii) $\mu^*(A) = \ell(A)$ if $A \in \mathcal{A}_0$;

(iii) every set in $\mathcal{A}_0$ is $\mu^*$-measurable;

(iv) if $\ell$ is $\sigma$-finite, then there is a unique extension to $\sigma(\mathcal{A}_0)$.  


Chapter 4

Measurable Functions

Definition 4.0.1. A function \( f : X \to \mathbb{R} \) is measurable or \( A \)-measurable if \( \{ x : f(x) > a \} \in A \) for all \( a \in \mathbb{R} \).

Proposition 4.0.2. Suppose \( f \) is real-valued. The following conditions are equivalent:

(i) \( \{ x : f(x) > a \} \in A \) for all \( a \in \mathbb{R} \);
(ii) \( \{ x : f(x) \leq a \} \in A \) for all \( a \in \mathbb{R} \);
(iii) \( \{ x : f(x) < a \} \in A \) for all \( a \in \mathbb{R} \);
(iv) \( \{ x : f(x) \geq a \} \in A \) for all \( a \in \mathbb{R} \);

Proposition 4.0.3. If \( X \) is a metric space, \( A \) contains all the open sets, and \( f : X \to \mathbb{R} \) is continuous, then \( f \) is measurable.

Proposition 4.0.4. Let \( c \in \mathbb{R} \). If \( f \) and \( g \) are measurable real-valued functions, then so are \( f + g, -f, cf, fg, \max(f, g), \) and \( \min(f, g) \).

Proposition 4.0.5. If \( f_i \) is a measurable real-valued function for each \( i \), then so are \( \sup_i f_i, \inf_i f_i, \lim \sup_{i \to \infty} f_i, \) and \( \lim \inf_{i \to \infty} f_i \).

Definition 4.0.6. We say \( f = g \) almost everywhere, written \( f = g \) a.e., if \( \{ x : f(x) \neq g(x) \} \) has measure zero. Similarly, we say \( f_i \to f \) a.e. if the set of \( x \) where \( f_i(x) \) does not converge to \( f(x) \) has measure zero.

If \( f : \mathbb{R} \to \mathbb{R} \) is measurable with respect to the Lebesgue \( \sigma \)-algebra, we say \( f \) is Lebesgue measurable.

Proposition 4.0.7. If \( f : \mathbb{R} \to \mathbb{R} \) is monotone, then \( f \) is Borel measurable hence Lebesgue measurable.

Proposition 4.0.8. Let \( (X, A) \) be a measurable space and let \( f : X \to \mathbb{R} \) be an \( A \)-measurable function. If \( A \) is in the Borel \( \sigma \)-algebra on \( \mathbb{R} \), then \( f^{-1}(A) \in A \).

Definition 4.0.9. Let \( (X, A) \) be a measurable space. If \( E \in A \), define the characteristic function of \( E \) by

\[
\chi_E(x) = \begin{cases} 
1, & x \in E; \\
0, & x \notin E.
\end{cases}
\]
A **simple function** \( s \) is a function of the form

\[
s(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)
\]

for real numbers \( a_i \) and measurable sets \( E_i \).

**Proposition 4.0.10.** Suppose \( f \) is a non-negative and measurable function. Then there exists a sequence of non-negative measurable simple functions \( s_n \) increasing to \( f \).

**Theorem 4.0.11.** (**Lusin’s Theorem**) Let \( f \) be a real-valued measurable function on \( E \). Then for each \( \epsilon > 0 \), there is a continuous function \( g \) on \( \mathbb{R} \) and a closed set \( F \) contained in \( E \) for which

\[
f = g \text{ on } F \text{ and } m(E - F) < \epsilon.
\]

**Theorem 4.0.12.** (**Egoroff’s Theorem**) Assume \( E \) has finite measure. Let \( \{f_n\} \) be a sequence of measurable functions on \( E \) that converge pointwise on \( E \) to the real-valued function \( f \). Then for each \( \epsilon > 0 \), there is a closed set \( F \) contained in \( E \) for which

\[
\{f_n\} \rightarrow f \text{ uniformly on } F \text{ and } m(E - F) < \epsilon.
\]
Chapter 5

The Lebesgue Integral

Definition 5.0.1. Let \((X, \mathcal{A}, \mu)\) be a measure space. If

\[ s = \sum_{i=1}^{n} a_i \chi_{E_i} \]

is a non-negative measurable simple function, define the Lebesgue integral of \(s\) to be

\[ \int s \, d\mu = \sum_{i=1}^{n} a_i \mu(E_i). \]

If \(f \geq 0\) is a measurable function, define

\[ \int f \, d\mu = \sup \left\{ \int s \, d\mu : 0 \leq s \leq f, \ s \text{ simple} \right\}. \]

Let \(f\) be measurable and let \(f^+ = \max(f, 0)\) and \(f^- = \max(-f, 0)\). So \(f = f^+ - f^-\) and \(|f| = f^+ + f^-\). Provided \(\int f^+ \, d\mu\) and \(\int f^- \, d\mu\) are not both infinite, define

\[ \int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu. \]

Definition 5.0.2. If \(f\) is measurable and \(\int |f| \, d\mu < \infty\), we say \(f\) is integrable.

Theorem 5.0.3. (Linearity and Monotonicity of Integration) Let the functions \(f\) and \(g\) be integrable over \(E\). Then for any \(a\) and \(b\), the function \(af + bg\) is integrable over \(E\) and

\[ \int_E (af + bg) = a \int_E f + b \int_E g. \]

Moreover,

\[ \text{if } f \leq g \text{ on } E, \text{ then } \int_E f \leq \int_E g. \]

The integral \(\int f \chi_A \, d\mu\) is often written \(\int_A f \, d\mu\).
Corollary 5.0.4. (Additivity Over Domains of Integration) Let $f$ be integrable over $E$. Assume $A$ and $B$ are disjoint measurable subsets of $E$. Then
\[ \int_{A \cup B} f = \int_A f + \int_B f. \]

Theorem 5.0.5. (The Countable Additivity of Integration) Let $f$ be integrable over $E$ and $\{E_n\}_{n=1}^\infty$ a disjoint countable collection of measurable subsets of $E$ whose union is $E$. Then
\[ \int_E f = \sum_{n=1}^\infty \int_{E_n} f. \]

Theorem 5.0.6. (The Continuity of Integration) Let $f$ be integrable over $F$.

(i) If $E_n \uparrow E$ is a countable collection of measurable subsets of $F$, then
\[ \int_E f = \lim_{n \to \infty} \int_{E_n} f. \]

(ii) If $E_n \downarrow E$ is a descending countable collection of measurable subsets of $F$, then
\[ \int_E f = \lim_{n \to \infty} \int_{E_n} f. \]

Theorem 5.0.7. Let $f$ be a bounded measurable function on a set of finite measure $E$. Then $f$ is integrable over $E$.

Proposition 5.0.8. If $f$ is integrable,
\[ \left| \int f \right| \leq \int |f|. \]

Proposition 5.0.9. Let the nonnegative function $f$ be integrable over $E$. Then $f$ is finite a.e. on $E$.

Proposition 5.0.10. Let $f$ be a nonnegative measurable function on $E$. Then
\[ \int_E f = 0 \text{ if and only if } f = 0 \text{ a.e. on } E. \]

Proposition 5.0.11. Let $f$ be integrable over $E$. Then $f$ is finite a.e. on $E$ and
\[ \int_E f = \int_{E - E_0} f \text{ if } E_0 \subset E \text{ and } m(E_0) = 0. \]
Chapter 6

Limit Theorems

Recall the following from calculus:

**Definition 6.0.1.** Let $f$ be a real-valued function defined on a set $E$ of real numbers. We say that $f$ is **continuous at** $x$ in $E$ provided that for each $\epsilon > 0$, there is a $\delta(\epsilon, x) > 0$ for which

$$
\text{if } x' \in E \text{ and } |x' - x| < \delta, \text{ then } |f(x') - f(x)| < \epsilon.
$$

The function $f$ is said to be **continuous** on $E$ provided it is continuous at each point in its domain $E$.

**Definition 6.0.2.** A real-valued function $f$ defined on a set $E$ of real numbers is said to be **uniformly continuous** provided for each $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that for all $x, x' \in E$,

$$
\text{if } |x - x'| < \delta, \text{ then } |f(x) - f(x')| < \epsilon.
$$

**Definition 6.0.3.** For a sequence $\{f_n\}$ of functions with common domain $E$, a function $f$ on $E$ and a subset $A$ of $E$, we say that

(i) The sequence $\{f_n\}$ converges **pointwise** on $A$ provided for each $x \in A$ and $\epsilon > 0$, there is an index $N(x, \epsilon)$ for which

$$
|f_n - f| < \epsilon \text{ on } A \text{ for all } n \geq N.
$$

(ii) The sequence $\{f_n\}$ converges to $f$ **pointwise a.e.** on $A$ provided it converges to $f$ pointwise on $A - B$, where $m(B) = 0$.

(iii) The sequence $\{f_n\}$ converges to $f$ **uniformly** on $A$ provided for each $\epsilon > 0$, there is an index $N(\epsilon)$ for which

$$
|f_n - f| < \epsilon \text{ on } A \text{ for all } n \geq N.
$$

(iv) For the sequence $\{f_n\}$ of measurable functions on $E$ and $f$ measurable for which $f$ and $f_n$ are finite a.e. on $E$, the sequence $\{f_n\}$ is said to converge in **measure** on $E$ to $f$ provided for each $\epsilon > 0$,

$$
\lim_{n \to \infty} m\{x \in E : |f_n(x) - f(x)| > \epsilon\} = 0
$$
Definition 6.0.4. A sequence of real numbers \( \{a_n\} \) is said to be Cauchy provided for each \( \epsilon > 0 \), there is an index \( N \) for which
\[
\text{if } n, m \geq N, \text{ then } |a_m - a_n| < \epsilon.
\]

Theorem 6.0.5. (The Extreme Value Theorem) A continuous real-valued function on a nonempty closed, bounded set of real numbers takes a minimum and maximum value.

Theorem 6.0.6. (The Intermediate Value Theorem) Let \( f \) be a continuous real-valued function on the closed, bounded interval \([a, b]\) for which \( f(a) < c < f(b) \). Then there is a point \( x_0 \in (a, b) \) at which \( f(x_0) = c \).

Theorem 6.0.7. (The Bolzano-Weierstrass Theorem) Every bounded sequence of real numbers has a convergent subsequence.

Theorem 6.0.8. (The Monotone Convergence Criterion) A monotone sequence of real numbers converges if and only it is bounded.

Theorem 6.0.9. (The Cauchy Convergence Criterion) A sequence of real numbers converges if and only if it is Cauchy.

Proposition 6.0.10. (i) Suppose \( \mu \) is a finite measure. If \( f_n \to f \) a.e., then \( f_n \) converges to \( f \) in measure. (ii) If \( \mu \) is a measure, not necessarily finite, and \( f_n \to f \) in measure, there is a subsequence such that \( f_{n_j} \to f \) a.e.

Lemma 6.0.11. (Chebyshev’s Inequality) If \( 1 \leq p < \infty \), then
\[
\mu(\{x : |f(x)| \geq a\}) \leq \frac{1}{a^p} \int |f|^p d\mu.
\]

Lemma 6.0.12. (Fatou’s Lemma) Let \( \{f_n\} \) be a sequence of nonnegative measurable functions on \( E \).
\[
\text{If } \{f_n\} \to f \text{ pointwise a.e. on } E, \text{ then } \int E \lim \inf f_n \leq \lim \inf \int E f_n.
\]

Proposition 6.0.13. Let \( \{f_n\} \) be a sequence of bounded measurable functions on a set of finite measure \( E \).
\[
\text{If } \{f_n\} \to f \text{ uniformly on } E, \text{ then } \lim_{n \to \infty} \int E f_n = \int E f.
\]

Theorem 6.0.14. (Monotone Convergence Theorem) Let \( \{f_n\} \) be an increasing sequence of nonnegative measurable functions on \( E \).
\[
\text{If } \{f_n\} \to f \text{ pointwise a.e. on } E, \text{ then } \lim_{n \to \infty} \int E f_n = \int E f.
\]

Theorem 6.0.15. (Dominated Convergence Theorem) Let \( \{f_n\} \) be a sequence of measurable functions on \( E \). Suppose there is a function \( g \) that is integrable over \( E \) and dominates \( \{f_n\} \) on \( E \) in the sense that \( |f_n| \leq g \) on \( E \) for all \( n \).
\[
\text{If } \{f_n\} \to f \text{ pointwise a.e. on } E, \text{ then } f \text{ is integrable over } E \text{ and } \lim_{n \to \infty} \int E f_n = \int E f.
\]
Chapter 7

Differentiation and Integration

Definition 7.0.1. A function $f$ is differentiable at $x$ if

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

exists, and the limit is called the derivative of $f$ at $x$ and denoted $f'(x)$. If $f : [a, b] \to \mathbb{R}$, we say $f$ is differentiable on $[a, b]$ if the derivative exists for each $x \in (a, b)$ and both

$$\lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h} \quad \text{and} \quad \lim_{h \to 0^-} \frac{f(b + h) - f(b)}{h}$$

exist.

Theorem 7.0.2. Let $f$ be a monotone function on the open interval $(a, b)$. Then $f$ is continuous except at a countable number of points in $(a, b)$.

Theorem 7.0.3. (Lebesgue’s Theorem) If the function $f$ is monotone on the open interval $(a, b)$, then it is differentiable almost everywhere on $(a, b)$.

Definition 7.0.4. Let $f$ be a real valued function defined on the closed, bounded interval $[a, b]$ and $P = \{x_0, \ldots, x_k\}$ be a partition $[a, b]$. Define the variation of $f$ with respect to $P$ by

$$V(f, P) = \sum_{i=1}^{k} | f(x_i) - f(x_{i-1}) | .$$

Definition 7.0.5. Let $f$ be a real-valued function defined on the closed, bounded interval $[a, b]$ and $P = \{x_0, \ldots, x_k\}$ be a partition of $[a, b]$. Define the total variation of $f$ with respect to $P$ by

$$TV(f) = \sup \{V(f, P) : P \text{ a partition of } [a, b]\} .$$

Definition 7.0.6. A real-valued function $f$ on the closed, bounded interval $[a, b]$ is said to be of bounded variation on $[a, b]$ provided

$$TV(f) < \infty$$
Corollary 7.0.7. If the function $f$ is of bounded variation on the closed, bounded interval $[a, b]$, then it is differentiable almost everywhere on the open interval $(a, b)$ and $f'$ is integrable over $[a, b]$.

Definition 7.0.8. A real-valued function $f$ on a closed, bounded interval $[a, b]$ is said to be absolutely continuous on $[a, b]$ provided for each $\epsilon > 0$, there is a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in $(a, b)$,

$$\text{if } \sum_{k=1}^n [b_k - a_k] < \delta \text{ then } \sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon.$$

Definition 7.0.9. A real-valued function $f$ is Lipschitz with constant $M$ if

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in \mathbb{R}$.

A function $f$ is Lipschitz with constant $M$ if and only if $f$ is absolutely continuous and $|f'| \leq M$ a.e.

Proposition 7.0.10. If the function $f$ is Lipschitz on a closed, bounded interval $[a, b]$, then it is absolutely continuous on $[a, b]$.

Theorem 7.0.11. Let the function $f$ be absolutely continuous on the closed, bounded interval $[a, b]$. Then $f$ is the difference of increasing absolutely continuous functions and, in particular, is of bounded variation.

Theorem 7.0.12. Let the function $f$ be absolutely continuous on the closed, bounded interval $[a, b]$. Then $f$ is differentiable almost everywhere on $(a, b)$, its derivative $f'$ is integrable over $[a, b]$ and

$$\int_a^b f' = f(b) - f(a).$$

Definition 7.0.13. We call a function $f$ on a closed, bounded interval $[a, b]$ the indefinite integral/antiderivative of $g$ over $[a, b]$ provided $g$ is Lebesgue integrable over $[a, b]$ and

$$f(x) = f(a) + \int_a^x g \text{ for all } x \in [a, b].$$

Theorem 7.0.14. A function $f$ on a closed, bounded interval $[a, b]$ is absolutely continuous on $[a, b]$ if and only if it is an indefinite integral over $[a, b]$.

Theorem 7.0.15. Let $f$ be integrable over the closed, bounded interval $[a, b]$. Then

$$\frac{d}{dx} \int_a^x f = f(x) \text{ for almost all } x \in (a, b).$$
CHAPTER 7. DIFFERENTIATION AND INTEGRATION

Definition 7.0.16. We say $f$ is **locally integrable** if $\int_K |f(x)| \, dx$ is finite whenever $K$ is compact.

Theorem 7.0.17. Let

$$f_r(x) = \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) \, dy.$$ 

If $f$ is locally integrable, then $f_r(x) \to f(x)$ a.e. as $r \to 0$.

Theorem 7.0.18. If $F : \mathbb{R} \to \mathbb{R}$ is increasing, then $F''$ exists a.e. and

$$\int_a^b F'(x) \, dx \leq F(b) - F(a)$$

whenever $a < b$. 

Chapter 8

Product Measures

Throughout this section \((X, \mathcal{A}, \mu)\) and \((Y, \mathcal{B}, \nu)\) are two reference measure spaces. Consider the Cartesian product \(X \times Y\) of \(X\) and \(Y\). If \(A \subset X\) and \(B \subset Y\), we call \(A \times B\) a rectangle. If \(A \in \mathcal{A}\) and \(B \in \mathcal{B}\), we call \(A \times B\) a measurable rectangle.

**Definition 8.0.1.** Let \(\mathcal{R}\) be the collection of measurable rectangles contained in \(X \times Y\), then \(\mu \times \nu\) is the product measure defined on \(\mathcal{R}\) by

\[
\mu \times \nu(A \times B) = \mu(A) \cdot \nu(B) \quad \text{for } A \times B \in \mathcal{R}.
\]

**Definition 8.0.2.** Let \(E\) be a subset of \(X \times Y\) and \(f\) a function on \(E\). For a point \(x \in X\) we define the \(x\)-section of \(E\) to be the set

\[
E_x = \{y \in Y : (x, y) \in E\} \subset Y
\]

and the function \(f(x, \cdot)\) defined on \(E_x\) by \(f(x, \cdot)(y) = f(x, y)\) to be the \(x\)-section of \(f\). The \(y\)-section is defined similarly.

**Theorem 8.0.3.** (Fubini-Tonelli Theorem) Suppose \(f : X \times Y \to \mathbb{R}\) is measurable with respect to \(\mathcal{A} \times \mathcal{B}\) if either (a) \(f\) is nonnegative, or (ii) \(\int |f(x, y)| \, d(\mu \times \nu)(x, y) < \infty\), then

(i) for each \(x\), the function \(y \mapsto f(x, y)\) is measurable with respect to \(\mathcal{B}\);
(ii) for each \(y\), the function \(x \mapsto f(x, y)\) is measurable with respect to \(\mathcal{A}\);
(iii) the function \(g(x) = \int f(x, y) \, \nu(dy)\) is measurable with respect to \(\mathcal{A}\);
(iv) the function \(h(y) = \int f(x, y) \, \mu(dx)\) is measurable with respect to \(\mathcal{B}\);
(v) we have

\[
\int f(x, y) \, d(\mu \times \nu)(x, y) = \int \left[ \int f(x, y) \, d\mu(x) \right] \, d\nu(y) = \int \left[ \int f(x, y) \, d\nu(y) \right] \, d\mu(dx).
\]
Chapter 9

Signed Measures

Definition 9.0.1. By a signed measure \( \nu \) on the measurable space \((X, \mathcal{M})\) we mean an extended real-valued set function \( \nu : \mathcal{M} \rightarrow [-\infty, \infty] \) that possesses the following properties:

(i) \( \nu \) assumes at most one of the values \( +\infty, -\infty \).
(ii) \( \nu(\emptyset) = 0 \).
(iii) For any countable collection \( \{E_k\}_{k=1}^{\infty} \) of disjoint measurable sets,

\[
\nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k).
\]

Definition 9.0.2. Let \( \nu \) be a signed measure. We say that a set \( A \) is positive (resp. negative) with respect to \( \nu \), provided \( A \) is measurable and for every subset \( E \) of \( A \) we have \( \nu(E) \geq 0 \) (resp. \( \nu(E) \leq 0 \)). A measurable set is called null with respect to \( \nu \) provided every measurable subset of it has \( \nu \) measure zero.

Theorem 9.0.3. (The Hahn Decomposition Theorem) Let \( \nu \) be a signed measure on the measurable space \((X, \mathcal{M})\). Then there is a positive set \( A \) for \( \nu \) and a negative set \( B \) for \( \nu \) for which \( X = A \cup B \) and \( A \cap B = \emptyset \).

Definition 9.0.4. A decomposition of \( X \) into the union of two disjoint sets \( A \) and \( B \) for which \( A \) is positive for \( \nu \) and \( B \) is negative is called a Hahn decomposition for \( \nu \). Such a decomposition may not be unique.

Definition 9.0.5. If \( \{A, B\} \) is a Hahn decomposition for \( \nu \), then we define two measures \( \nu^+ \) (the positive part) and \( \nu^- \) (the negative part) with \( \nu = \nu^+ - \nu^- \) by setting

\[
\nu^+(E) = \nu(E \cap A) \quad \text{and} \quad \nu^-(E) = -\nu(E \cap B).
\]

Definition 9.0.6. Two measures \( \nu_1 \) and \( \nu_2 \) on \((X, \mathcal{M})\) are said to be mutually singular (denoted \( \nu_1 \perp \nu_2 \)) if there are disjoint measurable sets \( A \) and \( B \) with \( X = A \cup B \) for which \( \nu_1(A) = \nu_2(B) = 0 \). The measures \( \nu^+ \) and \( \nu^- \) defined above are mutually singular.
Theorem 9.0.7. (The Jordan Decomposition Theorem) Let $\nu$ be a signed measure on the measurable space $(X, \mathcal{M})$. Then there are two mutually singular measures $\nu^+$ and $\nu^-$ on $(X, \mathcal{M})$ for which $\nu = \nu^+ - \nu^-$. Moreover, there is only one such pair of mutually singular measures.

Definition 9.0.8. The decomposition of a signed measure $\nu$ given by this theorem is called a Jordan decomposition of $\nu$.

Definition 9.0.9. The measure $|\nu| = \nu^+ + \nu^-$ is called the total variation measure of $\nu$. 
Chapter 10

$L^p$ Spaces

**Definition 10.0.1.** Let $X$ be a linear space. A real-valued functional $\| \cdot \|$ on $X$ is called a norm provided for each $f$ and $g$ in $X$ and each real number $a$,

$$
\|f + g\| \leq \|f\| + \|g\| \\
\|af\| = |a|\|f\| \\
\|f\| \geq 0 \text{ and } \|f\| = 0 \text{ if and only if } f = 0
$$

**Definition 10.0.2.** A normed linear space is a linear space together with a norm. If $X$ is a linear space normed by $\| \cdot \|$ we say that a function in $X$ is a unit function provided $\|f\| = 1$. For any $f \in X$, $f \neq 0$, the function $f/\|f\|$ is a unit function: it is a scalar multiple of $f$ which we call the normalization of $f$.

**Definition 10.0.3.** For $E$ measurable and for $1 \leq p < \infty$, we define $L^p(E)$ to be the collection of all functions $f$ for which

$$
\int_E |f|^p < \infty
$$

**Definition 10.0.4.** Let $(X, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space. For $1 \leq p < \infty$, define the $L^p$ norm of $f$ by

$$
\|f\|_p = \left( \int |f(x)|^p \, d\mu \right)^{\frac{1}{p}}
$$

**Definition 10.0.5.** We call a function $f$ essentially bounded on $E$, provided there is some $M \geq 0$, called an essential upper bound for $f$, for which

$$
|f(x)| \leq M \text{ for almost all } x \in E.
$$

**Definition 10.0.6.** For $E$ measurable, we define $L^\infty(E)$ to be the collection of all functions which are essentially bounded.

**Definition 10.0.7.** The $L^\infty$ norm of a function $f$ is the smallest $M$ such that $|f| \leq M$ a.e. Or,

$$
\|f\|_\infty = \inf \{ M : \mu(\{x : |f(x)| \geq M\}) = 0 \}$$
Definition 10.0.8. For \( 1 \leq p < \infty \), define the normed linear space \( \ell^p \) to be the collection of real sequences \( a = (a_1, a_2, \ldots) \) for which
\[
\sum_{k=1}^{\infty} |a_k|^p < \infty \text{ with norm being } \|a\|_p = \left( \sum_{k=1}^{\infty} |a_k|^p \right)^{\frac{1}{p}}.
\]
Similarly, we define the normed linear space \( \ell^\infty \) to be the collection of all real bounded sequences. For a sequence \( \{a_k\} \) in \( \ell^\infty \), define the norm to be
\[
\|\{a_k\}\|_\infty = \sup_{1 \leq k < \infty} |a_k|.
\]

Definition 10.0.9. The conjugate of a number \( p \in (1, \infty) \) is the number \( q = \frac{p}{p-1} \), which is the unique number \( q \in (1, \infty) \) for which
\[
\frac{1}{p} + \frac{1}{q} = 1.
\]
The conjugate of 1 is defined to be \( \infty \) and the conjugate of \( \infty \) is defined to be 1.

Lemma 10.0.10. (Young’s Inequality) For \( 1 < p < \infty \), \( q \) the conjugate of \( p \), and any two positive numbers \( a \) and \( b \),
\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}.
\]

Lemma 10.0.11. (Hölder’s Inequality) Let \( 1 < p, q < \infty \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Then for any Lebesgue measurable functions \( f \) and \( g \) on a Lebesgue measurable set \( E \), we have
\[
\int_E |f \cdot g| \leq \|f\|_p \cdot \|g\|_q.
\]

Lemma 10.0.12. (Minkowski’s Inequality) Let \( 1 \leq p < \infty \). Then for any Lebesgue measurable functions \( f \) and \( g \) on a Lebesgue measurable set \( E \), we have
\[
\|f + g\|_p \leq \|f\|_p + \|g\|_p.
\]

Corollary 10.0.13. Let \( E \) be a measurable set of finite measure and \( 1 \leq p_1 < p_2 \leq \infty \). Then \( L^{p_2}(E) \subset L^{p_1}(E) \).

Definition 10.0.14. A sequence \( \{f_n\} \) in a linear space \( X \) that is normed by \( \| \cdot \| \) is said to converge to \( f \) in \( X \) provided
\[
\lim_{n \to \infty} \|f_n - f\| = 0.
\]
We write
\[
\{f_n\} \to f \text{ or } \lim_{n \to \infty} f_n = f \text{ in } X
\]
to mean that each \( f_n \) and \( f \) belong to \( X \) and \( \lim_{n \to \infty} \|f_n - f\| = 0 \).
For a sequence \( \{f_n\} \) and function \( f \) in \( L^p(E) \), \( \{f_n\} \to f \) in \( L^p(E) \) if and only if
\[
\lim_{n \to \infty} \int_E |f_n - f|^p = 0.
\]

**Proposition 10.0.15.** If \( f_n \) converges to \( f \) in \( L^p \), then it converges in measure.

**Definition 10.0.16.** A sequence \( \{f_n\} \) in a linear space \( X \) that is normed by \( \| \cdot \| \) is said to be **Cauchy** in \( X \) provided for each \( \epsilon > 0 \), there is a natural number \( N \) such that
\[
\|f_n - f_m\| < \epsilon \text{ for all } m, n \geq N.
\]

A normed linear space \( X \) is said to be **complete** provided every Cauchy sequence in \( X \) converges to a function in \( X \). A complete normed linear space is called a **Banach space**.

**Proposition 10.0.17.** Let \( X \) be a normed linear space. Then every convergent sequence in \( X \) is Cauchy. Moreover, a Cauchy sequence in \( X \) converges if it has a convergent subsequence.

**Theorem 10.0.18.** (The Riesz-Fischer Theorem) Let \( E \) be a measurable set and \( 1 \leq p \leq \infty \). Then \( L^p(E) \) is a Banach space. Moreover, if \( \{f_n\} \to f \) in \( L^p(E) \), a subsequence of \( \{f_n\} \) converges pointwise a.e. on \( E \) to \( f \).

**Theorem 10.0.19.** Let \( E \) be a measurable set and \( 1 \leq p < \infty \). Suppose \( \{f_n\} \) is a sequence in \( L^p(E) \) that converges pointwise a.e. on \( E \) to the function \( f \) which belongs to \( L^p(E) \). Then
\[
\{f_n\} \to f \text{ in } L^p(E) \text{ if and only if } \lim_{n \to \infty} \int_E |f_n|^p = \int_E |f|^p.
\]
Chapter 11

The Radon-Nikodym Theorem

Definition 11.0.1. A measure $\nu$ is said to be absolutely continuous with respect to a measure $\mu$ if $\nu(A) = 0$ whenever $\mu(A) = 0$. We write $\nu \ll \mu$.

Theorem 11.0.2. (The Radon-Nikodym Theorem) Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space and $\nu$ a $\sigma$-finite measure defined on the measurable space $(X, \mathcal{M})$ that is absolutely continuous with respect to $\mu$. Then there is a nonnegative function $f$ on $X$ that is measurable with respect to $\mathcal{M}$ for which

$$\nu(E) = \int_E f \, d\mu \text{ for all } E \in \mathcal{M}.$$

The function $f$ is unique in the sense that if $g$ is any nonnegative measurable function on $X$ that also has this property, then $g = f$ a.e.

Theorem 11.0.3. (The Lebesgue Decomposition Theorem) Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite space and $\nu$ a $\sigma$-finite measure on the measurable space $(X, \mathcal{M})$. Then there is a measure $\nu_0$ on $\mathcal{M}$, singular with respect to $\mu$, and a measure $\nu_1$ on $\mathcal{M}$, absolutely continuous with respect to $\mu$, for which $\nu = \nu_0 + \nu_1$. The measures $\nu_0$ and $\nu_1$ are unique.
Chapter 12

Linear Operators

Definition 12.0.1. A linear functional on a linear space $X$ is a real-valued function $T$ on $X$ such that for $g$ and $h$ in $X$ and $a$ and $b$ real numbers,

$$T(a \cdot g + b \cdot h) = a \cdot T(g) + b \cdot T(h).$$

Definition 12.0.2. For a normed linear space $X$, a linear functional $T$ on $X$ is said to be bounded provided there is an $M \geq 0$ for which

$$|T(f)| \leq M \cdot \|f\| \text{ for all } f \in X.$$

The infimum of all such $M$ is called the norm of $T$ and denoted $\|T\|_*$. Also, $\|T\|_* = \sup \{T(f) : f \in X, \|f\| \leq 1\}$.

Definition 12.0.3. Let $X$ be a normed linear space. Then the collection of bounded linear functionals on $X$ is a linear space on which $\|\cdot\|_*$ is a norm. This normed linear space is called the dual space of $X$ and denoted $X^*$.

Proposition 12.0.4. Let $E$ be a measurable set, $1 \leq p < \infty$, $q$ the conjugate of $p$, and $g$ belong to $L^q(E)$. Define the functional $T$ on $L^p(E)$ by

$$T(f) = \int_E g \cdot f \text{ for all } f \in L^p(E).$$

Then $T$ is a bounded linear functional on $L^p(E)$ and $\|T\|_* = \|g\|_q$.

Theorem 12.0.5. (The Riesz Representation Theorem for the Dual of $L^p(E)$)

Let $E$ be a measurable set, $1 \leq p < \infty$, and $q$ the conjugate of $p$. For each $g \in L^q(E)$, define the bounded linear functional $\mathcal{R}_g$ on $L^p(E)$ by

$$\mathcal{R}_g(f) = \int_E g \cdot f \text{ for all } f \in L^p(E).$$

Then for each bounded linear functional $T$ on $L^p(E)$, there is a unique function $g \in L^q(E)$ for which

$$\mathcal{R}_g = T, \text{ and } \|T\|_* = \|g\|_q.$$
**Definition 12.0.6.** Let $X$ and $Y$ be linear spaces. A **linear operator** is a mapping $T : X \to Y$ provided for each $u, v \in X$, and real numbers $a$ and $b$,

$$T(au + bv) = aT(u) + bT(v).$$

**Definition 12.0.7.** Let $X$ and $Y$ be normed linear spaces. A linear operator $T : X \to Y$ is said to be **bounded** provided there is a constant $M \geq 0$ for which

$$\|T(u)\| \leq M\|u\| \text{ for all } u \in X.$$

Then infimum of all such $M$ is called the **operator norm** of $T$ and denoted $\|T\|$. The collection of bounded linear operators from $X$ to $Y$ is denoted by $\mathcal{L}(X,Y)$.

**Theorem 12.0.8.** A linear operator between normed linear spaces is continuous if and only if it is bounded.

**Proposition 12.0.9.** Let $X$ and $Y$ be normed linear spaces. Then the collection of bounded linear operators from $X$ to $Y$, $\mathcal{L}(X,Y)$, is a normed linear space.

**Theorem 12.0.10.** Let $X$ and $Y$ be normed linear spaces. If $Y$ is a Banach space, then so is $\mathcal{L}(X,Y)$.

**Definition 12.0.11.** A mapping $f : X \to Y$ from the topological space $X$ to the topological space $Y$ is said to be **open** provided the image of each open set in $X$ is open in the topological space $f(X)$.

**Theorem 12.0.12.** (The Open Mapping Theorem) Let $X$ and $Y$ be Banach spaces and the linear operator $T : X \to Y$ be continuous. Then its image $T(X)$ is a closed subspace of $Y$ if and only if the operator $T$ is open.

**Definition 12.0.13.** The **graph** of a mapping $T : X \to Y$ is the set $\{(x,T(x)) \in X \times Y : x \in X\}$. Therefore an operator is **closed** if and only if its graph is a closed subspace of the product space $X \times Y$.

**Theorem 12.0.14.** (The Closed Graph Theorem) Let $T : X \to Y$ be a linear operator between Banach spaces $X$ and $Y$. Then $T$ is continuous if and only if it is closed.

**Theorem 12.0.15.** (The Uniform Boundedness Principle) For $X$ a Banach space and $Y$ a normed linear space, consider a family $\mathcal{F} \subset \mathcal{L}(X,Y)$. Suppose the family $\mathcal{F}$ is pointwise bounded in the sense that for each $x$ in $X$ there is a constant $M_x \geq 0$ for which

$$\|T(x)\| \leq M_x \text{ for all } T \in \mathcal{F}.$$

Then the family $\mathcal{F}$ is uniformly bounded in the sense that there is a constant $M \geq 0$ for which $\|T\| \leq M$ for all $T$ in $\mathcal{F}$.

**Theorem 12.0.16.** (The Banach-Steinhaus Theorem) Let $X$ be a Banach space, $Y$ a normed linear space, and $\{T_n : X \to Y\}$ a sequence of continuous linear operators. Suppose that for each $x \in X$,

$$\lim_{n \to \infty} T_n(x) \text{ exists in } Y.$$
Then the sequence of operators \( \{T_n : X \to Y\} \) is uniformly bounded. Furthermore, the operator \( T : X \to Y \) defined by

\[
T(x) = \lim_{n \to \infty} T_n(x) \text{ for all } x \in X
\]

is linear, continuous, and

\[
\|T\| \leq \liminf \|T_n\|.
\]
Chapter 13

Hilbert Spaces

**Definition 13.0.1.** Let $H$ be a linear space. A function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ is called an inner product on $H$ provided for all $x_1, x_2, x$ and $y \in H$ and real numbers $a$ and $b$,

1. $\langle ax_1 + bx_2, y \rangle = a\langle x_1, y \rangle + b\langle x_2, y \rangle$,
2. $\langle x, y \rangle = \langle y, x \rangle$,
3. $\langle x, x \rangle > 0$ if $x \neq 0$.

A linear space $H$ together with an inner product on $H$ is called an inner product space.

**Lemma 13.0.2.** (The Cauchy-Schwarz Inequality) For any two vectors $u, v$ in an inner product space $H$,

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|.$$  

**Proposition 13.0.3.** For a vector $h$ in an inner product space $H$, define

$$\|h\| = \sqrt{\langle h, h \rangle}.$$  

Then $\| \cdot \|$ is a norm on $H$ called the norm induced by the inner product $\langle \cdot, \cdot \rangle$.

**Lemma 13.0.4.** (The Parallelogram Identity) For any two vectors $u, v$ in an inner product space $H$,

$$\|u - v\|^2 + \|u + v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$  

**Definition 13.0.5.** An inner product space $H$ is called a Hilbert space provided it is a Banach space with respect to the norm induced by the inner product.

**Definition 13.0.6.** Two vectors $u, v$ in the inner product space $H$ are said to be orthogonal provided $\langle u, v \rangle = 0$. A vector $u$ is said to be orthogonal to a subset $S$ of $H$ provided it is orthogonal to each vector in $S$. We denote by $S^\perp$ the collection of vectors in $H$ that are orthogonal to $S$.  

